AIAA JOURNAL
Vol. 33, No. 2, February 1995

# Integral Equation Formulation for Transient Structural Synthesis

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**An exact formulation for time domain structural synthesis is developed. Volterra integral equations are derived from the convolution integral which address substructure coupling and structural modification. The theory is cast in physical coordinates and, therefore, no transformation or mode truncation is required to achieve model reduction. As a minimum, only those coordinates directly involved in the synthesis need be retained, although synthesized transient response can be found for all coordinates, if so desired. The formulation makes use of transient response data and impulse response functions at the retained physical coordinates. Modified or coupled transient response is directly calculated; no synthesized system model is assembled. The formulation exactly synthesizes system damping, regardless of the uncoupled system damping models used. The numerical solution of the integral equations involves the solution of a lower triangular linear system only; no matrix factorization or eigensolution is required. Simple yet representative numerical examples are included.**

## Nomenclature

| | |
|---|---|
| $\boldsymbol{C}$ | = damping matrix |
| $f$ | = generalized excitation |
| $h, \boldsymbol{H}$ | = impulse response function, matrix |
| $\boldsymbol{K}$ | = stiffness matrix |
| $\boldsymbol{M}$ | = mass matrix |
| $t, \tau$ | = time |
| $\boldsymbol{R}$ | = equilibrium/compatibility matrix |
| $\boldsymbol{x}$ | = generalized dynamic response |
| $\alpha, \beta$ | = proportional damping coefficients |
| $\phi, \Phi$ | = mass normalized mode shape, modal matrix |
| $\omega, \Omega$ | = natural, circular frequency $s^{-1}$ |

*Subscripts*

| | |
|---|---|
| $c$ | = connection coordinates |
| $d$ | = damped system quantity |
| $h$ | = homogeneous solution |
| $i$ | = internal coordinates, mode index |

*Superscripts*

| | |
|---|---|
| $*$ | = synthesized or change |
| $\sim$ | = coupling quantity |
| $\dot{}, \ddot{}$ | = time derivatives |

## Introduction

SUBSTRUCTURE coupling and structural modification (collectively referred to as structural synthesis) are concerned with the calculation of dynamic response for a structural system obtained from the coupling of two or more substructures together, and/or from the addition or removal of structural elements from the system. A primary motivation for such analyses is the computational economy obtained by synthesizing a system model which is smaller than the sum of the component models and, in many cases, is smaller than each of the component models. An additional motivation (among others) for structural synthesis is that information about the system is often known only at the substructure level. For example, vibration testing for model validation is typically performed on individual substructures and, hence, damping is known only for the substructures.

The most common methods for substructure coupling are referred to collectively as component mode synthesis (CMS), see, for example, Ref. 1. These methods represent each substructure by a set of suitably chosen physical and modal coordinates and their associated shapes. These methods achieve a reduction in model order by truncating the number of modal coordinates retained for each substructure. The individual substructure models so reduced are assembled yielding a set of differential equations which must then be solved, requiring an additional eigensolution and subsequent calculation of physical response. A disadvantage of these methods is that the accuracy of the synthesized system model is related to the model order; to obtain a more accurate system model, more substructure coordinates need to be retained, hence increasing the order of the system model. Furthermore, although damping can be synthesized using CMS, the relationship between substructure damping (which is often known accurately from a modal test) and the system damping is not known.

An alternative method for structural synthesis which circumvents the aforementioned disadvantages is frequency domain structural synthesis.[2-4] Using frequency response functions (FRF) calculated for the substructures, FRF for the coupled system are directly and exactly calculated. This method operates with physical coordinate system descriptions and, therefore, can provide exact coupled system frequency response using, as a minimum, only interface coordinates. Of course, noninterface coordinate frequency response can be synthesized as well. Advantages of this method include the arbitrary order reduction of the presynthesis frequency response function model available simply by extracting the FRF elements of interest; the synthesis is always exact regardless of the dimension of the reduction. Furthermore, the method synthesizes damping exactly, regardless of the nature of the damping in the substructure models. A limitation of this method is that it does not directly treat transient problems.

The motivation for the present work is to provide a direct and exact time domain structural synthesis method which provides the benefits of the physical coordinate, frequency domain method, summarized as follows: 1) exact model reduction simply by retaining only the physical coordinates of interest, and 2) exact synthesis of damping. Furthermore, the theory we will develop eliminates the assembly and solution of the synthesized system equations; the theory provides for the direct calculation of synthesized system transient response based on arbitrary presynthesis transient response. In fact, the governing equations to be derived provide an analytic relationship between the synthesized transient response and the presynthesis system parameters and transient response. We will consider, first, structural modification.

## Theory: Structural Modification

Consider an arbitrary structure whose physical coordinates $\{\boldsymbol{x}\}$ are partitioned into $\{\boldsymbol{x}_c\}$ and $\{\boldsymbol{x}_i\}$. The $\{\boldsymbol{x}_c\}$ are those coordinates at which a structural modification is to be installed (connection

Received March 30, 1994; revision received Oct. 26, 1994; accepted for publication Oct. 26, 1994. Copyright © 1994 by the American Institute of Aeronautics and Astronautics, Inc. All rights reserved.

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coordinates), such as the nodal coordinates at which an additional element is being added. The $\{x_i\}$ are the physical coordinates of the structure not directly involved in the modification (internal coordinates), but about which postmodification (synthesized) transient response information is desired.

The total dynamic response of the system can be written in terms of the convolution integral as follows:

$$\begin{Bmatrix} x_i(t) \\ x_c(t) \end{Bmatrix} = \begin{Bmatrix} x_i(t) \\ x_c(t) \end{Bmatrix}_h + \int_0^t \begin{bmatrix} H_{ii}(t-\tau) & H_{ic}(t-\tau) \\ H_{ci}(t-\tau) & H_{cc}(t-\tau) \end{bmatrix} \begin{Bmatrix} F_i(t) \\ F_c(t) \end{Bmatrix} d\tau \tag{1}$$

The $H(t)$ are matrices of impulse response functions. The subscript $h$ indicates the homogeneous solution containing the constants of integration. This Eq. (1) is the basis for both structural modification and substructure coupling.

Continuing with the development of the governing equation for transient structural modification, the structure experiences only externally applied forces at the coordinate set $i$, and the both externally applied forces and reaction forces imposed by the modification at the coordinate set $c$, i.e.,

$$\begin{Bmatrix} F_i(\tau) \\ F_c(\tau) \end{Bmatrix} = \begin{Bmatrix} F_i^e(\tau) \\ F_c^e(\tau) \end{Bmatrix} + \begin{Bmatrix} 0 \\ F_c^*(\tau) \end{Bmatrix} \tag{2}$$

The superscript $e$ denotes externally applied and the superscript * denotes a quantity associated with the modified system. Introducing Eq. (2) into Eq. (1) yields

$$\begin{Bmatrix} x_i(t) \\ x_c(t) \end{Bmatrix}^* = \begin{Bmatrix} x_i(t) \\ x_c(t) \end{Bmatrix}_h + \int_0^t \begin{bmatrix} H_{ii}(t-\tau) & H_{ic}(t-\tau) \\ H_{ci}(t-\tau) & H_{cc}(t-\tau) \end{bmatrix} \times \left\{ \begin{Bmatrix} F_i^e(\tau) \\ F_c^e(\tau) \end{Bmatrix} + \begin{Bmatrix} 0 \\ F_c^*(\tau) \end{Bmatrix} \right\} d\tau \tag{3}$$

where the * superscript on the left-hand side indicates a synthesized quantity, i.e., the transient response of the structure with the modification installed. Equation (3) can be rewritten as

$$\begin{Bmatrix} x_i(t) \\ x_c(t) \end{Bmatrix}^* = \begin{Bmatrix} x_i(t) \\ x_c(t) \end{Bmatrix} + \int_0^t \begin{bmatrix} H_{ic}(t-\tau) \\ H_{cc}(t-\tau) \end{bmatrix} \{F_c^*(\tau)\} d\tau \tag{4}$$

because we recognize that

$$\begin{Bmatrix} x_i(t) \\ x_c(t) \end{Bmatrix} = \begin{Bmatrix} x_i(t) \\ x_c(t) \end{Bmatrix}_h + \int_0^t \begin{bmatrix} H_{ii}(t-\tau) & H_{ic}(t-\tau) \\ H_{ci}(t-\tau) & H_{cc}(t-\tau) \end{bmatrix} \begin{Bmatrix} F_i^e(\tau) \\ F_c^e(\tau) \end{Bmatrix} d\tau$$

The vector $\{F_c^*(t)\}$ represents the reaction forces imposed by the structural modification on the host structure and can be written generally as

$$\{F_c^*(t)\} = -[M^*]\{\ddot{x}^*(t)\} - [C^*]\{\dot{x}^*(t)\} - [K^*]\{x^*(t)\} \tag{5}$$

Extracting the second row of Eq. (4) and introducing Eq. (5) yields the governing equation for structural modification,

$$\{x_c^*(t)\} = \{x_c(t)\} - \int_0^t [H_{cc}(t-\tau)]\{[M^*]\{\ddot{x}^*(\tau)\} - [C^*] \times \{\dot{x}^*(\tau)\} - [K^*]\{x^*(\tau)\}\} d\tau \tag{6}$$

Equation (6) is a nonstandard nonhomogeneous Volterra integrodifferential equation (VIDE) of the second kind. The unknown vector $\{x_c^*(t)\}$ represents the transient response of the modified system, and it is this vector for which Eq. (6) must be solved. For problems involving a few physical coordinates, Eq. (6) can be solved using Laplace transforms. For higher order systems, a numerical solution is required. Rather than pursuing the solution of a nonstandard VIDE, we integrate Eq. (6) by parts twice to obtain

$$\begin{aligned} &[[I] + [\dot{H}_{cc}(0)][M^*]]\{x_c^*(t)\} = \{x_c(t)\} \\ &\quad + [[\dot{H}_{cc}(t)][M^*] + [H_{cc}(t)][C^*]]\{x_0\} \\ &\quad + [H_{cc}(t)][M^*]\{\dot{x}_0\} - \int_0^t [[\ddot{H}_{cc}(t-\tau)][M^*] \\ &\quad + [\dot{H}_{cc}(t)][C^*] + [H_{cc}(t)][K^*]]\{x_c^*(t)\}\, d\tau \end{aligned} \tag{7}$$

where we have used $\{x^*(0)\} = \{x(0)\} = \{x_0\}$ and $\{\dot{x}^*(0)\} = \{\dot{x}(0)\} = \{\dot{x}_0\}$, which can be seen from Eq. (6). Equation (7) is a standard linear Volterra nonhomogeneous integral equation of the second kind.[5]

Equation (7) possesses unique and continuous solutions because[5,6]

$$[[I] + [\dot{H}_{cc}(0)][M^*]]^{-1}\{\{x_c(t)\} + \cdots$$

$$[[\dot{H}_{cc}(t)][M^*] + [H_{cc}(t)][C^*]]\{x_0\} + [H_{cc}(t)][M^*]\{\dot{x}_0\}\}$$

is continuous in $0 \le t \le T$,

$$[[\ddot{H}_{cc}(t-\tau)][M^*] + [\dot{H}_{cc}(t)][C^*] + [H_{cc}(t)][k^*]]$$

is continuous in $0 \le \tau \le t \le T$, and

$$[[\ddot{H}_{cc}(t-\tau)][M^*] + [\dot{H}_{cc}(t)][C^*] + [H_{cc}(t)][k^*]]$$

satisfies the Lipschitz condition for all $0 \le \tau \le t \le T$. For zero initial conditions, Eq. (7) reduces to

$$\begin{aligned} [[I] + [\ddot{H}_{cc}(0)][M^*]]\{x_c^*(t)\} = \{x_c(t)\} - \int_0^t [[\ddot{H}_{cc}(t-\tau)][M^*] \\ + [\ddot{H}_{cc}(t)][C^*] + [H_{cc}(t)][k^*]]\{x_c^*(t)\}\, d\tau \end{aligned} \tag{8}$$

Once Eq. (7) has been solved for the solution $\{x_c(t)\}^*$, the transient responses of the internal coordinates are calculated using the top row of Eq. (4), along with any necessary finite difference approximations for the derivatives of $\{x_c(t)\}^*$, as required by Eq. (5).

We now derive the governing integral equations for substructure coupling.

## Theory: Substructure Coupling

We begin again with Eq. (1) where $\{x_c\}$ is a vector of physical coordinates directly involved in the coupling, and $\{x_i\}$ are coordinates not directly involved but about which postsynthesis information is desired. Note that for the coupling of two substructures $a$ and $b$ the coordinate sets are partitioned as

$$\{x_c(t)\} = \lfloor x_c^a \quad x_c^b \rfloor^T \qquad \{x_i(t)\} = \lfloor x_i^a \quad x_i^b \rfloor^T$$

The substructures experience only externally applied forces at their coordinate sets $i$, and both externally applied forces and reaction forces imposed by the attaching substructure at the coordinate sets $c$, hence, introducing Eq. (2) into Eq. (1) and following the same process as in the structural modification problem, we are led to Eq. (4) once again, where the vector $\{F_c^*(t)\}$ represents the coupling forces between substructures. We specify coupling force equilibrium through the Boolean matrix

$$\{F_c^*(t)\} = [R]\{\tilde{F}_c^*(t)\} \tag{9}$$

where each column of $[R]$ is all zeros except for a 1 and a $-1$ in the rows corresponding to the coordinates being coupled together. The vector $\{F_c^*(t)\}$ represents the independent coupling force between each pair of coordinates. Introducing Eq. (9) into the second row of Eq. (4) and premultiplying by $[R]^T$ yields

$$\{\tilde{x}_c^*(t)\} = \{\tilde{x}_c(t)\} + \int_0^t [\tilde{H}_{cc}(t-\tau)]\{\tilde{F}_c^*(\tau)\}\, d\tau \tag{10}$$

where

$$\{\tilde{x}_c(t)\} = [R]^T\{x_c(t)\} \quad \text{and} \quad [\tilde{H}_{cc}(t)] = [R]^T[H_{cc}(t)][R]$$

We specify compatibility between the mutual coupling coordinates by the condition

$$\{\tilde{x}_c^*(t)\} = [R]^T\{x_c^*(t)\} = \{0\}$$

This yields a homogeneous linear Volterra integral equation of the first kind[5] for the coupling force vector $\{\tilde{F}_c^*(t)\}$,

$$\{\tilde{x}_c^*(t)\} = -\int_0^t [\tilde{H}_{cc}(t-\tau)]\{\tilde{F}_c^*(\tau)\}\,d\tau \tag{11}$$

Note that Eq. (11) requires that $\{\tilde{x}_c(0)\} = \{0\}$, with $\|[\tilde{H}_{cc}(t-\tau)]\|$ bounded, for the solution to be bounded.[6]

We may pursue a numerical solution of Eq. (11), but we will first convert Eq. (11) into a second-kind equation by differentiation. Taking two derivatives provides

$$[\dot{\tilde{H}}_{cc}(0)]\{\tilde{F}_c^*(t)\} = -\{\ddot{\tilde{x}}_c^*(t)\} - \int_0^t [\ddot{\tilde{H}}_{cc}(t-\tau)]\{\tilde{F}_c^*(\tau)\}\,d\tau \tag{12}$$

where a unique continuous solution for $\{\tilde{F}_c^*(t)\}$ is guaranteed because[6]

$$[\ddot{\tilde{H}}_{cc}(t-\tau)] \text{ and } [\dot{\tilde{H}}_{cc}(0)]$$

are continuous in $0 \le \tau \le t \le T$, $[\dot{\tilde{H}}_{cc}(0)]$ does not vanish anywhere in $0 \le t \le T$, and $\{\tilde{x}_c(0)\} = \{0\}$ and $\{\tilde{x}_c(t)\}$ and its derivatives are continuous in $0 \le t \le T$. This Eq. (12) is solved numerically for the coupling force vector. This is substituted into the first row of Eq. (4) which is directly integrated to solve for the synthesized transient response.

## Numerical Solution of Volterra Integral Equation

The numerical solution of Volterra equations is a topic that has received much attention (e.g., Refs. 5–10). Many of the numerical methods common in the solution of differential equations have been applied to integral equations. In fact, integral equations are often converted into differential equations and then solved. Here, we will pursue two simple schemes for the direct solution of the governing equations for modification and coupling, which are both Volterra equations of the second kind. The schemes are based on replacing the integrals in Eqs. (7), (8), and (12) with a simple quadrature rule, to be demonstrated by example.

### Example 1. Stiffness Modification

As an example of the numerical solution of a structural modification problem, consider the stiffness modification of an undamped single degree-of-freedom (DOF) system, shown in Fig. 1. The synthesis is performed using the transient response of the original spring/mass system along with its impulse response function. The transient response to a sinusoidal input $F \sin \Omega t$ is

$$x(t) = \frac{F}{\omega^2 - \Omega^2}\left(\sin \Omega t - \frac{\Omega}{\omega}\sin \omega t\right)$$

and the impulse response is

$$h(t) = \sin \omega t/\omega$$

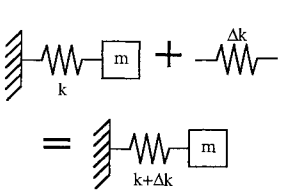


**Fig. 1 Stiffness modification of a simple oscillator.**

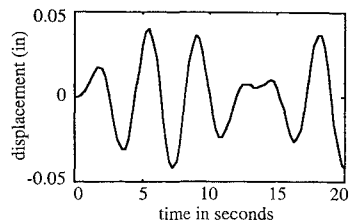


**Fig. 2 Transient response of the spring-mass system prior to stiffness modification.**

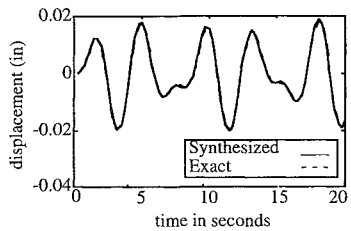


**Fig. 3 Synthesized and exact modified transient response; the two curves are identical with respect to the resolution of the plot.**

The transient response prior to modification is plotted in Fig. 2. The governing equation (7) for modification reduces to

$$x^*(t) = x(t) - \frac{\Delta K}{\omega}\int_0^t x^*(t)\sin \omega(t-\tau)\,d\tau \tag{13}$$

The transient response will be calculated at the time $t_i = i\Delta t$; $i = 0, 1, 2 \ldots n$. Letting $L_{ij} = \sin \omega(i\Delta t - j\Delta t)$, the integral in Eq. (13) can be replaced, using the trapezoidal rule[5] by

$$x^*(t_i) = x(t_i) - \frac{\Delta k}{\omega}\Delta t\left[\frac{1}{2}L_{i0}x^*(0\Delta t) + \sum_{j=1}^{i-1} L_{ij}x^*(j\Delta t) + \frac{1}{2}L_{i0}x^*(i\Delta t)\right] \qquad i = 0, 1, 2, \ldots, n$$

These $n+1$ equations constitute a lower triangular linear system,

$$\{x\} = [L]\{x^*\} \tag{14}$$

where $\boldsymbol{L}$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{\Delta k\Delta t}{2\omega}L_{10} & 1 & \vdots & \vdots & \vdots & 0 \\ \vdots & \frac{\Delta k\Delta t}{\omega}L_{21} & 1 & \vdots & \vdots & 0 \\ \vdots & \vdots & \frac{\Delta k\Delta t}{\omega}L_{32} & 1 & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & 1 & 0 \\ \frac{\Delta k\Delta t}{2\omega}L_{n0} & \cdots & \cdots & \cdots & \frac{\Delta k\Delta t}{\omega}L_{n(n-1)} & 1 \end{bmatrix}$$

Equation (14) is solved by forward substitution. Note that the trapezoid rule has first-order accuracy and does not require starting values. The lower triangular structure results because $L_{ij} = 0$ for $i \le j$. If $\omega = 19.66\ \text{s}^{-1}$, $F = 1$ lbf, and $\Delta k = 40$ lbf/in., $\Delta t = 0.25$ s ($\Delta t \approx (2\pi/10\omega^*)$, and $n = 80$ for 20 s, the solution shown in Fig. 3 is obtained.

### Example 2. Substructure Coupling

An example of the coupling of two substructures is presented, based on the use of Eq. (12). Each substructure is a proportionally damped cantilever beam modeled using two elements each. See the Appendix for the substructure model parameters and element matrices. The two substructures, prior to coupling and after, are shown in Fig. 4. Each beam element has 2 DOF per node, a lateral translation and a rotation. Therefore, referring to Fig. 4, the two

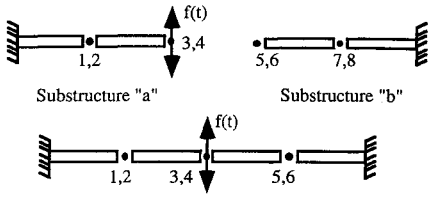


**Fig. 4 Two cantilever beam finite element models, before and after coupling.**

systems are coupled through DOFs 3–5 and DOFs 4–6. Substructure $a$ is subjected to a transient excitation,

$$f(t) = \begin{cases} P_0 \sin \Omega t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

and we will solve for the transient response of DOF 5 in the coupled system. Note that the DOF in the $c$ set for substructure $a$ are 3 and 4, and those for substructure $b$ are 1 and 2.

The transient response of substructure $a$ along with the impulse response models of both substructures are calculated using the respective system modes (the transient response of substructure $b$ is zero). The total response for a mode $i$ is

$$q_i(t) = q_{hi}(t) + \int_0^t h_i(t-\tau)\{\phi\}^T\{f(\tau)\}\,d\tau$$

where

$$q_{hi}(t) = e^{-\zeta_i \omega_i t}[A_i \cos(\omega_{di} t) + B_i \sin(\omega_{di} t)]$$

The $\{\phi\}$ is a mass normalized mode shape and

$$h_i(t) = \frac{1}{\omega_{di}} e^{-\zeta \omega_i t} \sin(\omega_{di} t)$$

where the subscript $h$ indicates the homogeneous solution, and the subscript $d$ indicates a damped natural frequency.

With all initial conditions set to zero (as required by the first kind form of the governing integral equation for coupling) the modal response reduces to

$$q_i(t) = \int_0^t h_i(t-\tau)\phi^T f(\tau)\,d\tau$$

The physical impulse response matrix $\boldsymbol{H}(t)$ for a structural system can be found from the modal impulse response functions $h_i(t)$,

$$[H(t)] = \begin{bmatrix} \boldsymbol{\Phi}^a & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Phi}^b \end{bmatrix} \begin{bmatrix} \text{diag}[h^a(t)] & \mathbf{0} \\ \mathbf{0} & \text{diag}[h^b(t)] \end{bmatrix} \begin{bmatrix} \boldsymbol{\Phi}^a & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Phi}^b \end{bmatrix}^T$$

Equation (12) requires the $(c, c)$ partitions associated with the connection coordinates to be extracted from the impulse response matrix and two derivatives taken to produce $[\ddot{\boldsymbol{H}}_{cc}(t)]$. Note also that

$$[\dot{\boldsymbol{H}}_{cc}(0)] = \left[\boldsymbol{\Phi}^a_{cc}\right]\left[\boldsymbol{\Phi}^a_{cc}\right]^T + \left[\boldsymbol{\Phi}^b_{cc}\right]\left[\boldsymbol{\Phi}^b_{cc}\right]^T$$

for mass normalized mode shapes.

Equation (12) is used to perform the coupling. Here we will replace the integral in Eq. (12) using Simpson's one-third rule for those points in time arrived at over an even number of time intervals. For those points in time arrived at over an odd number of intervals, we use Simpson's one-third rule up to the last four points, and then we use Simpson's three-eighth rule for the last four points. The use of the three-eighth rule at the end of the interval rather than at the beginning is crucial for computational reasons.[6,7] The integral in Eq. (12) is, therefore, approximated as

$$\int_0^{t_i} [\ddot{\boldsymbol{H}}_{cc}(t-\tau)]\{\tilde{\boldsymbol{F}}^*_c(\tau)\}\,dt \approx \Delta t \sum_{j=0}^{i} W_{ij}[\ddot{\boldsymbol{H}}_{cc}(i\Delta t - j\Delta t)] \times \{\tilde{\boldsymbol{F}}^*_c(j\Delta t)\}$$

where $t_i = i\Delta_t$, and the $w_{ij}$ are the weights consistent with the already described quadrature based on Simpson's rule. We again arrive at a lower triangular linear system which is solved for the synthesized system transient coupling forces.

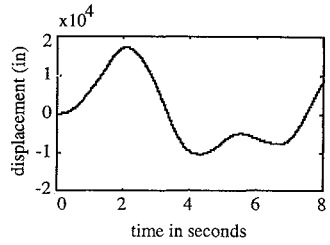


**Fig. 5 Uncoupled transient response of DOF 3.**

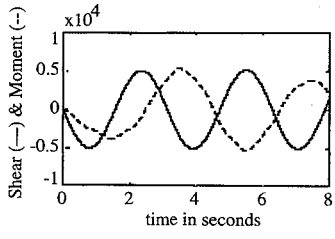


**Fig. 6 Generalized coupling forces $\{\tilde{f}(t)\}$.**

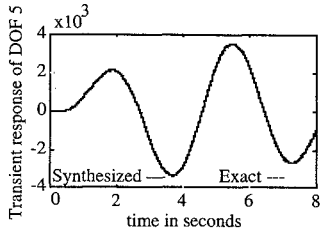


**Fig. 7 Synthesized and exact transient response calculated for DOF 5; the two curves are identical with respect to the resolution of the plot.**

Note that the use of higher order quadrature formulas, such as Simpson's rule, requires starting values. For Simpson's rule, two starting values are required.[11] Note also that the convergence of the numerical method is relatively insensitive to errors in the starting values; the effect of the starting errors on convergence is attenuated by the factor $\Delta t$ (Ref. 6).

The uncoupled transient response of DOF 3 is shown in Fig. 5. Figure 6 shows the coupling force and moment which is the solution of Eq. (12). Figure 7 compares the exact solution for the coupled transient response with that calculated using the integral equation formulation. The solution was calculated using a time step approximately two orders of magnitude smaller than the shortest period expected for the coupled system.

## Discussion

The time domain method developed herein is closely related to frequency domain structural synthesis.[2–4] The governing equation for frequency domain synthesis is

$$[Y]^* = [Y] - [Y_{ec}][R]\left[\tilde{Z}^{-1} + \tilde{Y}_{cc}\right]^{-1}[R]^T[Y_{ce}]$$

and $[\tilde{Y}_{cc}] = [R]^T[Y_{cc}][R]$. Here, the $[Y]$ are frequency response function matrices. As mentioned at the outset, the advantages of the frequency domain method for structural modification and substructure coupling motivated the development of the present method. The advantages that these methods share include the following.

1) Arbitrary model reduction is available. As the methods are cast in physical coordinates, only those coordinates of interest need be retained.

2) There is an exact synthesis of damping. Any linear damping model may be included. The method is exact because it treats the coupling/modification forces explicitly.

3) No synthesized system model is developed and, therefore, no additional solution phase is required.

These physical coordinate methods, in fact, constitute analytic relations for the synthesized response as functions of the presynthesis subsystem parameters and response.

The development of the time domain formulation continues. Efficient numerical schemes for large-scale computation need to be identified along with practical error bounds. One error bound arises out of a proof of convergence. It is shown in Ref. 6 that the error is proportional to $e^{WLT}$, where $W$ is the maximum quadrature weight, $T$ the final value of time, and $L$ the Lipschitz constant such that, and with respect to Eq. (12), for example,

$$[\ddot{\tilde{H}}_{cc}(t,\tau)]\left\{\{\tilde{F}_c^*(t)\}^2 - \{\tilde{F}_c^*(t)\}^1\right\} \leq L\left\{\{\tilde{F}_c^*(t)\}^2 - \{\tilde{F}_c^*(t)\}^1\right\}$$

and $L$ may be taken as

$$L = \left\| \left[\Phi_{cc}^a\right]\left[\Phi_{cc}^a\right]^T + \left[\Phi_{cc}^b\right]\left[\Phi_{cc}^b\right]^T \right\|$$

However, this bound is very crude and does not provide useful information with respect to maximum step size required for an accurate solution. This is an issue of current work. Experience has shown that a step size two orders of magnitude smaller than the shortest natural period in the problem is required for the quadrature rules employed here. The frequencies of the coupled system can be determined by performing the following determinant search:

$$\det\left[\tilde{\mathbf{Z}}^{-1} + \tilde{Y}_{cc}\right] = \mathbf{0}$$

where $\mathbf{Z}$ is the impedance matrix corresponding to Eq. (5). The zero crossings of this frequency dependent equation occur at the natural frequencies of the synthesized system. The substructure coupling example involved proportionally damped beams. Although the theory exactly treats any linear damping model, a generally effective numerical method of solution must be so demonstrated, which is another topic of further work.

## Appendix: Finite Element Model Data

Each substructure comprises two 4-DOF beam elements. The element mass and stiffness matrices are[12]

$$[k_e] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & \text{sym} & 12 & -6L \\ & & & 4L^2 \end{bmatrix}$$

$$[m_e] = \frac{m}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ & 4L^2 & 13L & -3L^2 \\ & \text{sym} & 156 & -6L \\ & & & 4L^2 \end{bmatrix}$$

and $EI = L = m/420 = 1$.

Proportional damping is assigned as

$$[C] = \alpha[K] + \beta[M]$$

with $\alpha = 0.0$, and $\beta = 2.0$. The natural frequencies for the two substructures are

$$[\omega] = 2\pi \text{ diag}(0.14, 0.88, 2.99, 8.68)\ \text{s}^{-1}$$

and the natural frequencies for the coupled system are

$$[\omega] = 2\pi \text{ diag}(1.39, 3.85, 7.63, 14.57, 24.10, 38.79)\ \text{s}^{-1}$$

The magnitude of the excitation $P_0 = 100.0$ lbf.

## Acknowledgments

The author is indebted to Clyde Scandrett, Department of Mathematics, Naval Postgraduate School, for his substantial insight most generously given.

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